Indian Statistical Institute
Final Examination 2017-2018
M.Math First Year
Functional Analysis

Time : 3 Hours Date : 11.05.2018 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Note: $\mathcal{B}(\cdots)$ = the set of all bounded linear operators.

Q1. (10 marks) Is the following statement true or false (with justification)? - "c_0 is reflexive".

Q2. (10 marks) Give an example of an operator $T \in \mathcal{B}(l^2)$ such that $T \neq 0$ but $\sigma(T) = \{0\}$.

Q3. (10 marks) Let X be a normed linear space and $\mathcal{S}$ a subspace of X. Prove that exactly one of the following holds:
(i) $\mathcal{S}$ is dense in X; (ii) there exists a non-zero $\varphi \in X^*$ such that $\mathcal{S} \subseteq \ker \varphi$.

Q4. (10 marks) If $(A, \mathcal{A}, \mu)$ is a σ-finite measure space and $f \in L^\infty(A, \mathcal{A}, \mu)$, then prove that

$$\|M_f\|_{\mathcal{B}(L^2(A,\mathcal{A},\mu))} = \text{ess sup} f.$$

Q5. (15 marks) Prove that every finite-dimensional subspace of a normed linear space is complemented.

Q6. (15 marks) Let X be a normed linear space and $\{x_n\} \subseteq X$. If $\{\varphi(x_n)\}$ is bounded for all $\varphi \in X^*$, then prove that $\{x_n\}$ is bounded.

Q7. (15 marks) Let X be an infinite-dimensional Banach space and $S, T \in \mathcal{B}(X)$. Which of the following statements are true (with justification)?
(i) If $T^2 = 0$, then T is compact.
(ii) If ST is compact, then either S or T is compact.
(iii) If $T^m = I$ for some $m \geq 1$, then T is not compact.

Q8. (15 marks) Let $\{e_n\}_{n\geq 1}$ be the standard orthonormal basis of l^2. Define $T : l^2 \to l^2$ by

$$Te_n = \frac{1}{n} e_{n+1},$$

for all $n \geq 1$. Prove that T is compact and $\sigma_{point}(T) = \emptyset$. Is this a contradiction to the spectral theorem for compact and self-adjoint operators? Justify your answer.

Q9. (15 marks) If $\mathcal{H}$ is an infinite-dimensional and separable Hilbert space, then prove that the closed unit ball centered at the origin of $\mathcal{B}(\mathcal{H})$ is not separable in the norm topology.